Fluid Dynamics of Relativistic Quantum Dust

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Abstract

The microscopic transport equations for free fields are solved using the Schwinger function. Thus, for general initial conditions, the evolution of the energy-momentum tensor is obtained, incorporating the quantum effects exactly. The result for relativistic fermions differs from classical hydrodynamics, which is illustrated for Landau and Bjorken type initial conditions in this model of exploding primordial matter. Free fermions behave like classical dust concerning hydrodynamic observables. However, quantum effects which are present in the initial state are preserved.
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Often the complicated time dependent dynamics of quantum many-body systems or fields is approximated by a perfect fluid model. Since the seminal work by Fermi and Landau this approach has been applied successfully, in order to study global features, such as multiplicity distributions and apparently thermal transverse momentum spectra of produced particles, in high-energy collisions of strongly interacting matter [1, 2, 3, 4]. Similarly, the hydrodynamic approximation is often invoked in astrophysical applications and cosmological studies of the early universe [5]

Recently it has been shown that a free scalar field indeed behaves like a perfect fluid in the semiclassical (WKB) regime [6]. More generally, the mechanisms of quantum decoherence and thermalization in such systems which can be described hydrodynamically, i.e. the emergence of classical deterministic evolution from an underlying quantum field theory, are of fundamental interest [7, 8, 9, 10].

The limitations of the fluid picture, however, have rarely been explored in the microscopic or high energy density domain. Difficulties reside in the derivation of consistent transport equations and in the amount of computation required to find realistic solutions; see Refs. [11], for example, for a review and recent progress concerning selfinteracting scalar particles and the quark-gluon plasma, respectively. More understanding of related hydrodynamic behavior, if any, seems highly desirable.

Presently, we study the relation between relativistic hydrodynamics and the full quantum evolution of a free matter field. In the absence of interactions, decoherence or thermalization may be present in the initial state, corresponding to an impure density matrix, but is followed by unitary evolution. We consider this as a "quantum dust" model of the expansion of matter originating from a high energy density preparation phase, which the Landau and Bjorken models describe classically [2, 3].

Our approach is independent of the nature of the field, as long as it obeys a standard wave equation. To be definite, we choose to work with Dirac fermions and comment about neutrinos later. We introduce the spinor Wigner function, i.e., a (4x4)-matrix depending on space-time and four-momentum coordinates:

$$W_{\alpha\beta}(x;p) \equiv \int \frac{\mathrm{d}^4 y}{(2\pi)^4} e^{-ip\cdot y} \langle : \bar{\psi}_\beta(x+y/2)\psi_\alpha(x-y/2) : \rangle \quad , \tag{1}$$

where the expectation value refers to the (mixed) state of the system; without interactions, the vacuum plays only a passive role and, therefore, is eliminated by normal-ordering the field operators.

All observables can be expressed in terms of the Wigner function here. In particular, the (unsymmetrized) energy-momentum tensor:

$$\langle : T_{\mu\nu}(x) : \rangle \equiv i \langle : \bar{\psi}(x) \gamma_\mu \overset{\leftrightarrow}{\partial}_\nu \psi(x) : \rangle = \mathrm{tr}\ \gamma_\mu \int \mathrm{d}^4 p\ p_\nu W(x;p) \quad , \tag{2}$$

where $\overset{\leftrightarrow}{\partial} \equiv \frac{1}{2}(\overrightarrow{\partial} - \overleftarrow{\partial})$ and with a trace over spinor indices (conventions as in [12]). Furthermore, the dynamics of W reduces to the usual phase space description in the classical limit [12].

Propagation of the free fields entering in Eq. (1) from one time-like hypersurface to another is described by the Schwinger function. It is the solution of the homogeneous Dirac equation, $[i\gamma \cdot \partial_x - m] S(x,x') = 0$, for the initial condition $S(\vec{x},\vec{x}', x^0 = x'^0) = -i\gamma^0 \delta^3(\vec{x} - \vec{x}')$. Thus, $\psi(x) = i \int \mathrm{d}^3 x' S(x,x') \gamma^0 \psi(x')$, and similarly for the adjoint. An explicit form is:

$$iS(x,x') = iS(x - x' \equiv \Delta) = (i\gamma \cdot \partial_\Delta + m) \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2\omega_k} \left(e^{-ik_+ \cdot \Delta} - e^{-ik_- \cdot \Delta} \right) \quad , \tag{3}$$

where $k_\pm \equiv (\pm\omega_k, \vec{k})$ and $\omega_k \equiv (\vec{k}^2 + m^2)^{1/2}$.

Making use of Eqs. (1) and (3), we relate the Wigner function at different times, $t = x^0, x'^0$:

$$W(x;p) = \int \frac{\mathrm{d}^4 k}{(2\pi)^3} e^{-ik\cdot x} \delta^\pm(p_k^+) \delta^\pm(p_k^-) \int \mathrm{d}^3 x' e^{ik\cdot x'} \int \mathrm{d}p'^0 \Lambda(p_k^+) \gamma^0 W(x';p') \gamma^0 \Lambda(p_k^-) \quad , \tag{4}$$

where $p_k^\pm \equiv p \pm \frac{k}{2}$, $\delta^\pm(q) \equiv \pm\delta(q^2 - m^2)$ (for $q^0/|q^0| = \pm 1$), $\Lambda(q) \equiv \gamma \cdot q + m$, and $p'^\mu \equiv (p'^0, \vec{p})$.

The Eq. (4) implies that the Wigner function obeys a generalized mass-shell constraint and a proper free-streaming transport equation:

$$[p^2 - m^2 - \frac{\hbar^2}{4} \partial_x^2]\ W(x;p) \quad = \quad 0 \quad , \tag{5}$$

$$p \cdot \partial_x\ W(x;p) \quad = \quad 0 \quad , \tag{6}$$

separately for each matrix element. The reinserted $\hbar$ indicates the important quantum term in the equations, which otherwise have the familiar classical appearance.

Thus Eq. (4) presents an integral solution of the microscopic transport equations for a given initial Wigner function. Furthermore, a semiclassical approximation of the Schwinger function may be used to generate an integral solution of the corresponding classical transport problem.

Next, we decompose the Wigner function with respect to the standard basis of the Clifford algebra, $W = \mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^\mu \gamma^5 \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} \mathcal{S}_{\mu\nu}$, i.e., in terms of scalar, pseudoscalar, vector, axial vector, and antisymmetric tensor components. The functions, $\mathcal{F} \equiv \frac{1}{4} \mathrm{tr}\ W$, $\mathcal{P} \equiv -\frac{1}{4} i\ \mathrm{tr}\ \gamma^5 W$, $\mathcal{V}_\mu \equiv \frac{1}{4} \mathrm{tr}\ \gamma_\mu W$, $\mathcal{A}_\mu \equiv \frac{1}{4} \mathrm{tr}\ \gamma_5 \gamma_\mu W$, and $\mathcal{S}_{\mu\nu} \equiv \frac{1}{4} \mathrm{tr}\ \sigma_{\mu\nu} W$, which represent physical current densities, are real, due to $W^\dagger = \gamma^0 W \gamma^0$ [12]. They individually obey Eqs. (5) and (6).

We assume $\mathcal{P} = 0 = \mathcal{A}_\mu$, i.e., we consider a spin saturated system for simplicity [13]. Then, using the 'transport equation' which follows directly from the Dirac equation applied to W, $[\gamma \cdot (p + \frac{i}{2} \partial_x) - m] W(x;p) = 0$, and decomposing it accordingly, the following additional relations among the remaining densities are obtained:

$$\mathcal{V}^\mu(x;p) \quad = \quad \frac{m p^\mu}{p^2} \mathcal{F}(x;p) \quad , \tag{7}$$

$$\mathcal{S}^{\mu\nu}(x;p) \quad = \quad \frac{1}{2p^2} (p^\nu \partial_x^\mu - p^\mu \partial_x^\nu) \mathcal{F}(x;p) \quad . \tag{8}$$

Note that $\mathcal{S}^{\mu\nu}$ is intrinsically by one order in $\hbar$ smaller than the other two densities.

We conclude that presently the dynamics of the system is represented completely by the scalar phase space density $\mathcal{F}$. Using Eqs. (2) and (7), we obtain in particular:

$$\langle : T^{\mu\nu}(x) : \rangle = 4m \int \mathrm{d}^4p \, \frac{p^\mu p^\nu}{p^2} \mathcal{F}(x;p) \ , \tag{9}$$

which is symmetric and conserved, $\partial_\mu T^{\mu\nu}(x) = 0$, on account of Eq. (6). Furthermore, this implies the 'equation of state':

$$\langle : T^{00}(x) : \rangle - \sum_{i=1}^{3} \langle : T^{ii} : \rangle = 4m \int \mathrm{d}^4p \, \mathcal{F}(x;p) \ , \tag{10}$$

which relates energy density and pressure(s). However, applying Eq. (5), we find that this relationship evolves in a wavelike manner, driven by off-shell contributions to the evolving $\mathcal{F}$:

$$\partial_x^2 \langle : T^\mu_\mu(x) : \rangle = 16m \int \mathrm{d}^4p \, (p^2 - m^2)\mathcal{F}(x;p) \ . \tag{11}$$

This differs from classical hydrodynamics with a fixed functional form of the equation of state. Eqs. (9)-(11) hold independently of the initial state, of course, if it evolves without further interaction.

Making use of Eq. (4) in Eq. (9), we now calculate the energy-momentum tensor at any time in terms of the initial scalar density. Employing the decomposition of the Wigner function and commutation and trace relations for the γ matrices, as well as Eqs. (5)-(8), we obtain:

$$\begin{aligned} \langle : T^{\mu\nu}(x) : \rangle = 8m \int \dots \int \frac{p^\mu p^\nu}{p^2} \Big(& p^2 + \frac{m^2}{p'^2} \left(p^0 p'^0 + \vec{p}^{\,2} \right) - \frac{1}{4p'^2} \Big((p^0 k^0)^2 - \vec{p}^{\,2} \vec{k}^2 \\ & + p^0 p'^0 k^2 + \frac{p^0}{p'^0} \, (k^0)^2 p^2 \Big) \Big) \, \mathcal{F}(x'; \vec{p}, p'^0) \ , \end{aligned} \tag{12}$$

where $\int \dots \int \equiv (2\pi)^{-3} \int \mathrm{d}^4p \int \mathrm{d}^4k \; e^{-ik\cdot x} \delta^\pm(p_k^+) \delta^\pm(p_k^-) \int \mathrm{d}^3x' e^{ik\cdot x'} \int \mathrm{d}p'^0$; we also made use of partial integrations and the δ-function constraints. The three terms on the right-hand side stem from the scalar, vector, and antisymmetric tensor components of the initial Wigner function, respectively.

If the initial distribution is an isotropic function of the three-momentum, then $T^{\mu\nu}$ is diagonal at all times, implying that the absence of flow in the initial state will be preserved.

Indeed, we expect the (non-)flow features of the initial distribution to be preserved during the evolution, due to the absence of interactions. Kinetic energy from microscopic particle degrees of freedom will not be converted into collective motion. An interesting question is, how the classical hydrodynamic acceleration of fluid cells due to pressure gradients arises in our present model after coarse graining [7, 8, 9, 10]. We do not pursue this at present. Recalling earlier work on the hydrodynamic representation of quantum mechanics, e.g. Refs. [14], and recently deduced classical fluid behavior of quantum fields in WKB approximation [6], we study the full quantum effects here.

We consider the exact evolution of $T^{\mu\nu}$, assuming a particle-antiparticle symmetric initial state. This is believed to hold, for example, close to midrapidity in the center-of-mass frame of central high-energy collisions [3, 4]. It implies that the initial $\mathcal{F}$ is an even function of the energy variable, $\mathcal{F}(x'; \vec{p}, p'^0) = \mathcal{F}(x'; \vec{p}, -p'^0)$. While Eq. (12) allows general initial conditions, we follow the implicit on-shell assumption in classical hydrodynamic models:

$$\mathcal{F}(x'; \vec{p}, p'^0) = (2\pi)^{-3} m \; \delta(p'^2 - m^2) \left(\Theta(p'^0) F(x'; \vec{p}, p'^0) + \Theta(-p'^0) F(x'; \vec{p}, -p'^0) \right) \ . \tag{13}$$

Fermion blackbody radiation is described by $F(x'; \vec{p}, p'^0) \equiv f(p'^0/T(x'))$, where T denotes the local temperature, and with $f(s) \equiv (e^s + 1)^{-1}$; this is easily illustrated with the help of Eqs. (9) and (13).

Implementing Eq. (13), we obtain the simpler result:

$$\langle : T^{\mu\nu}(x) : \rangle = \int \frac{\mathrm{d}^3x'\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\;\frac{p^\mu p^\nu \cos[\vec{k}\cdot(\vec{x}-\vec{x}')]}{\omega_p\omega_+\omega_-}F(x';\vec{p},\omega_p) \tag{14}$$
$$\cdot\{((\omega_+ +\omega_-)^2-\vec{k}^2)\cos[(\omega_+ -\omega_-)t]-((\omega_+ -\omega_-)^2-\vec{k}^2)\cos[(\omega_+ +\omega_-)t]\}\;,$$

where $t\equiv x^0-x'^0$, $\omega_p\equiv(\vec{p}^2+m^2)^{1/2}$, $\omega_\pm\equiv((\vec{p}\pm\vec{k}/2)^2+m^2)^{1/2}$; furthermore, $p^0\equiv\frac{1}{2}|\omega_+\pm\omega_-|$, with "+" when multiplying the first and "−" when multiplying the second term of the difference, respectively. Depending on geometry and initial state, further integrations can be done analytically.

Consider a (1+1)-dimensional system for illustration, assuming that the particles are approximately massless, i.e. $\omega_p\approx|p|$, and that F is even in p (no flow). Specializing to a Landau type initial condition, the distribution is prepared on a fixed timelike hypersurface at $t=0$ [2]. We find the ultrarelativistic equation of state for the only nonvanishing components of $T^{\mu\nu}$, $\epsilon\equiv T^{00}=T^{11}\equiv P$ ($d=1+1$), which are calculated as a momentum integral following Eq. (14):

$$T^{00}(x,t) = 2\int\frac{\mathrm{d}p}{2\pi}|p|\Big(F(x-t;|p|)+F(x+t;|p|)\Big) \tag{15}$$
$$= \frac{1}{2}\Big(T^{00}(x-t,t=0)+T^{00}(x+t,t=0)\Big)\;, \tag{16}$$

i.e., a superposition of wavelike propagating momentum contributions in accordance with Eq. (11).

Similarly, a Bjorken type initial condition can be specified on a surface of constant proper time [3]. A transformation of Eq. (15) to space-time rapidity and proper time coordinates yields:

$$T^{00}(y,\tau)=2\int\frac{\mathrm{d}p}{2\pi}|p|\Big(F(-\tau_0 e^{-y/2+\ln\tau/\tau_0};|p|)+F(\tau_0 e^{y/2+\ln\tau/\tau_0};|p|)\Big)\;, \tag{17}$$

since $x\equiv\tau\sinh y/2$ and $t\equiv\tau\cosh y/2$ ($\tau\geq\tau_0>0$).

Our results for free fermions show the free-streaming behavior of *classical dust*, associated with the independent propagation and linear superposition of the momentum contributions to the scalar component F of the Wigner function here. In particular, the shape function of each mode is preserved and translated lightlike (with dispersion for massive particles). Due to the assumed momentum symmetry, the initial distribution will separate into two components after a finite time, travelling into the forward and backward direction, respectively, with a corresponding dilution at the center.

We recall that $T^{\mu\nu}$ being diagonal implies the absence of ideal hydrodynamic flow, given $\epsilon=(d-1)P$. This does not depend on whether the initial state is on- or off-shell, see Eq.(12). Therefore, any hydrodynamic behavior must be the effect of a pecularity of the semiclassical limit [6], of coarse graining [8, 9, 10], or of interactions [15], or a combination of these.

Despite the apparently classical evolution, however, all initial state quantum effects are incorporated and preserved. If the initial dimensionless distribution F has a dependence on products of momentum and space-time variables, which is characteristic for matter waves, such terms invoke a factor $1/\hbar$. Similarly, if it is thermal (T) but includes the finite size (L) shell effects or global constraints, then there are quantum corrections involving $LT/\hbar$ ($k_B=c=1$) [16]. They have not been included in semiclassical tranport or classical hydrodynamic models of high-energy (nuclear) collisions, but may be large. Here the quantum dust model provides a testing ground to assess the importance of these quantum effects.

Our approach based on the Schwinger function reduces the solution of the free quantum transport problem to quadratures – analytical results in three dimensions can be obtained and will be discussed elsewhere. It may lead to an efficient way of treating interacting particles, when a perturbative expansion is meaningful.

References

[1] E. Fermi, Progr. Theor. Phys. **5**, 570 (1950); Phys. Rev. **81**, 683 (1951).

[2] L. D. Landau, Izv. Akad. Nauk SSSR, Ser. fiz., **17**, 51 (1953); S. Z. Belenkij and L. D. Landau, N. Cim., Suppl., **3**, 15 (1956).

[3] J. D. Bjorken, Phys. Rev. **D27**, 140 (1983).

[4] See also the proceedings of the International Conferences on Ultra-Relativistic Nucleus-Nucleus Collisions; e.g., *Quark Matter '99*, eds. L. Riccati, M. Masera and E. Vercellini (Elsevier, Amsterdam, 1999).

[5] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[6] G. Domenech and M. L. Levinas, Physica **A278**, 440 (2000).

[7] W. Zurek (ed.), *Complexity, Entropy and the Physics of Information* (Addison-Wesley, Reading, Mass., 1990).

[8] H.-Th. Elze, Nucl. Phys. **B436**, 213 (1995); Phys. Lett. **B369**, 295 (1996); [quant-ph/9710063].

[9] A. M. Lisewski, *On the classical hydrodynamic limit of quantum field theories*, [quant-ph/9905014].

[10] T. A. Brun and J. B. Hartle, Phys. Rev. **D60**, 123503 (1999).

[11] J.-P. Blaizot and E. Iancu, Nucl. Phys. **B557**, 183 (1999); D. Boyanovsky, F. Cooper, H. J. de Vega and P. Sodano, Phys. Rev. **D58**, 025007 (1998); H.-Th. Elze and U. Heinz, Phys. Rep. **183**, 81 (1989).

[12] D. Vasak, M. Gyulassy and H.-Th. Elze, Ann. Phys. (N.Y.) **173**, 462 (1987).

[13] From this point on, a corresponding study of (approximately) massless Standard Model $\nu_L\bar{\nu}_R$ neutrinos differs, and is simpler, since $\mathcal{V}_\mu = \mathcal{A}_\mu$, while all other densities vanish identically; see, e.g., H.-Th. Elze, T. Kodama and R. Opher, Phys. Rev. **D63** (2001) 013008.

[14] E. Madelung, Z. Phys. **40** (1926) 322; G. Holzwarth and D. Schütte, Phys. Lett. **B73** (1978) 255; S. K. Ghosh and B. M. Deb, Phys. Rep. **92** (1982) 1.

[15] L. M. A. Bettencourt, F. Cooper and K. Pao, *Hydrodynamic scaling from the dynamics of relativistic quantum field theory*, [hep-ph/0109108].

[16] H.-Th. Elze and W. Greiner, Phys. Lett. **B179**, 385 (1986); Phys. Rev. **A33**,1879 (1986).